Uncertainty in seismic inverse problems

Background

Seismic reflection data is used to estimate the properties of the Earth's subsurface from reflected seismic waves. The method is similar to sonar, echolocation, and medical imaging. One main task for a geophysicist is to convert the seismic response recorded on the surface to a 3D representation of the subsurface. Achieving the best possible quality of image and estimation of rock properties is critical in helping identify new reservoirs and understand the depletion and sweep mechanisms in existing reservoirs. Even marginal improvements here can significantly reduce the uncertainty associated with an opportunity, helping to improve our return on investment.

Inversion of multi-dimension of seismic data is typically achieved by a local optimization scheme, such as steepest gradient descent, during which the difference between modelled and observed seismic data (L2 norm) is minimized by iteratively updating the earth parameters (ie. Velocities, density, etc.). Such optimization method only yields a deterministic outcome and requires high data quality to succeed. To better quantify the uncertainty in earth parameter estimation, a probabilistic approach is needed. Bayesian techniques may be used to produce a posterior distribution that captures a range of possible earth models, thus providing a probabilistic outcome to the seismic inverse problem.

An MCMC algorithm

The Bayes theorem for our problem can be written as:

$$p(\boldsymbol{m}|\boldsymbol{d}^g) \propto p(\boldsymbol{m})p(\boldsymbol{d}^g|\boldsymbol{m})$$

where $\boldsymbol{m}$ is earth parameters sampled in depth, and $\boldsymbol{d}^g$ is seismic data in a gather sampled in time.

By assuming a Gaussian prior and observation errors, the prior and likelihood can be written as:

$$P(\boldsymbol{m}) = e^{\left[-\frac{1}{2}(\boldsymbol{m}-\mu_m^d)\,\Sigma_m^{-1}(\boldsymbol{m}-\mu_{\boldsymbol{m}}^d)^T\right]}$$

$$P(\boldsymbol{d}|\boldsymbol{m}) = e^{\left[-\frac{1}{2}(\boldsymbol{d}^g-f(\boldsymbol{m}))\,\Sigma_d^{-1}(\boldsymbol{d}^g-f(\boldsymbol{m}))^T\right]}$$

where μ_m^d is the Gaussian prior mean, and $f(\boldsymbol{m})$ is the simulated seismic data. The operator f is a non-linear operator that acts on $\boldsymbol{m}$ and returns simulated data $\boldsymbol{d}_{sim} = f(\boldsymbol{m})$.

The posterior can be assessed by a random walk Metropolis-Hastings algorithm with the acceptance ratio

$$a = \min\left[\frac{p(\boldsymbol{m}_i|\boldsymbol{d}^g) \times q(\boldsymbol{m}_{i-1}|\boldsymbol{m}_i)}{p(\boldsymbol{m}_{i-1}|\boldsymbol{d}^g) \times q(\boldsymbol{m}_i|\boldsymbol{m}_{i-1})}, 1\right]$$

Due to the dimensionality of the problem, convergence of this MCMC algorithm is typically very slow. So we need to find ways to sample the posterior more efficiently.

Current solution

Making 'better' proposals

Often the proposal distributions are chosen to be symmetric (e.g. Gaussian, t-distribution). We can attempt to find an approximate posterior *P** that can be used as the proposal distribution. By simplifying the physics of the forward problem, consider the revised acceptance ratio as follows:

$$a = \min\left[\frac{p(\boldsymbol{m}_i|\boldsymbol{d}^g)p^*(\boldsymbol{m}_{i-1}|\boldsymbol{d}^s)}{p(\boldsymbol{m}_{i-1}|\boldsymbol{d}^g)p^*(\boldsymbol{m}_i|\boldsymbol{d}^s)}, 1\right]$$

where $\boldsymbol{d}^s$ is a stacked seismic trace rather than a gather of traces. To simulate $\boldsymbol{d}^s$ from the earth model $\boldsymbol{m}$, a linear operator f^* is used,

$$f^* = C \times D$$

where C is a seismic wavelet and D is a differential operator that converts earth parameters to a reflectivity series, and the simulated data $\boldsymbol{d}^s_{sim} = f^*(\boldsymbol{m})$. The approximate posterior $P^*(\boldsymbol{m}|\boldsymbol{d})$ is Gaussian,

$$P^*(\boldsymbol{m}|\boldsymbol{d}) = N\left(\boldsymbol{m}; \mu^*_{\boldsymbol{m}|\boldsymbol{d}}, \Sigma^*_{\boldsymbol{m}|\boldsymbol{d}}\right)$$

and the mean $\mu^*_{\boldsymbol{m}|\boldsymbol{d}}$ and covariance $\Sigma^*_{\boldsymbol{m}|\boldsymbol{d}}$ can be found by:

$$\Sigma_{\boldsymbol{d},\boldsymbol{m}} = f^*\Sigma_{\boldsymbol{m}}$$

$$\Sigma_{\boldsymbol{m},\boldsymbol{d}} = \Sigma^T_{\boldsymbol{d},\boldsymbol{m}}$$

$$\Sigma_{\boldsymbol{m}|\boldsymbol{d}} = \Sigma_{\boldsymbol{m}} - \Sigma_{\boldsymbol{m},\boldsymbol{d}}\Sigma^{-1}_{\boldsymbol{d}}\Sigma^T_{\boldsymbol{m},\boldsymbol{d}}$$

$$\mu_{\boldsymbol{m}|\boldsymbol{d}} = \mu_{\boldsymbol{m}}\Sigma^{-1}_{\boldsymbol{d}}\left(\boldsymbol{d}^s - f^*(\mu_{\boldsymbol{m}})\right)$$

where $\mu_{\boldsymbol{m}}$ and $\Sigma_{\boldsymbol{m}}$ are the prior mean and covariance, respectively.

Python code has been written for the above algorithm, but it has not been fully tested yet. There are some numerical issues encountered with the computation of the approximate posterior that need to be addressed. These numerical issues may be related to how the prior is specified, so prior model construction is also something that needs to be explored further.

Thoughts for some alternative solutions

Hamiltonian MC – Improves slow exploration of the parameter space by making proposals to the Metropolis-Hastings algorithm by considering Hamiltonian dynamics. The HMC makes use of the momentum variables to augment the target distribution. This introduces the computation of the kinetic energy associated with the state, in addition to the potential energy.

Reverse Jump MCMC - There are often many possible earth model, with different dimensions of the parameter space. Multi-model inference techniques like RJMCMC allows samples to be drawn from the posterior by jumping between different models. This is an attractive feature for seismic inversion because the subsurface structure is made up of layers of different thicknesses.

Hybrid MCMC – Combine the advantage of sampling efficiency of HMC with the desirable feature of multi-model inference offered by RJMCMC.

Approximate Posterior Inference – Variational methods. Posterior inherence is transformed into an optimization problem, where a variational distribution is introduced to approximate the actual posterior.